Many Roads from the Axiom of Completeness[*]

Andrew Nightingale

Abstract

  "We all grow up believing in the existence of real numbers . . ." (Abbot, 2001, p. 244). The "property that distinguishes" the real numbers is the Axiom of Completeness (Abbot, 2001, p. 244). The Axiom is a dry mathematical statement, or collection of equivalent statements. Jorge Luís Borges, the blind seer of metaphor (the dead root of every word), can help resurrect the dead metaphors behind the Axiom. It turns out the Axiom represents a wishful sort of thinking called logical induction. The desire behind induction is knowledge of the unknown, in the abstract. Does the Axiom give us knowledge of the unknown; does it solve ancient problems such as motion and being? Exploring what the Axiom means and what metaphors it hides releases a vertigo of ideas that swirl and coalesce into an inquiry into thinking of "the question" in itself. Imagining numbers without the grave Axiom leads one to wonder: Could an entirely different world of levity and wonder emerge?

Jorge Luis Borges, in a lecture at Harvard, raised the possibility "that all metaphors are made by linking two different things together . . ." ("The Metaphor," 1967). His definition is circular because for any metaphor (as in "my heart is the sun") the "is" sets up a relation of equality ("heart" equals "sun"), making metaphor the relationship "is." What then "is" metaphor? In mathematics, "=" cannot stand with just one thing ("3 ="), even though "=" is often translated into English as "is."[1] Borges says metaphor is circular by quoting the poet Lugones, a Spaniard who lived in Argentina the same time as Borges: "'Every

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[1] The law of Identity makes "A = A," but here "=" is between two separate "A's","A=" is not used.

word is a dead metaphor.' This of course is a metaphor" (Borges, 1967). Words are made of metaphors, and metaphors are made of words. Even the verb "to be" is a dead metaphor of something else. "To be" grew from the Proto-Indo-European "bheue" which also meant "to grow" and from "bheue" branched out into "bhumih"— "the world" in Sanskrit. One tries to speak simply of things being, but doing so links them to other things using dead metaphors. Borges offered the Chinese world of "the 10,000 things" and, by limiting himself to "metaphors that we feel *are* metaphors," he explored "stock" metaphors, which "appear in all literatures." Some metaphors repeat themselves over and over, which suggests a world of only "10,000 things" is possible. If one could only talk about things without the compounding effect of metaphors, would there be only 10,000 words, one for each thing, or would there be the well-explored labyrinth of infinity in mathematics?

Borges talks about one of the "stock" poetic metaphors: "We are such stuff. As dreams are made on" (Shakespeare, *The Tempest*, Act 4, Scene 1, 155–156). Here he notices a "slight contradiction" and asks, "If we are real in dreams, or if we are merely dreamers of dreams, then I wonder if we can make such sweeping statements." Mathematics does make "sweeping statements." In a world of metaphor, can these statements wake us up to the things themselves, things with a non-metaphorical being? Do we have the power to imagine our way out of the imagination, to perceive an awakening? Perhaps numbers have that power to take us through metaphor to the real things. Some consider the problem already settled, and one mathematician even said we are in an "era in which the real numbers are all-powerful" (Hartshorne, 2000, p. 3).

Before being sifted and refined into "real" ones, numbers are spoken, written and meant the same as words. The words "real numbers" mean that the numbers are real and not metaphorical, which entangles them in a complex relationship between Sign and Object. Why are they real and not metaphorical? How can anyone make the claim "numbers are real" without the implied metaphor of that statement?

The philosopher of science Paul Feyerabend argues that what a number *is* changed dramatically before Pythagoras's first revelation that "everything is number" and the painstaking work of proving the revelation that went on for millennia, almost to this day. "Examining

archaic number concepts, many researchers assume that they share basic elements. Tables of addition, subtraction, multiplication, which are 'correct' when interpreted in modern terms, seem to confirm the assumption. But such an interpretation cannot explain why . . . the oldest known number systems were dyadic and why this feature survived right into the beginning of arithmetic" (Feyerabend, 1999, p. 88).

The story of how numbers made their progress toward reality is a myth that other fields share as well. Feyerabend continues by quoting Vasari, the first Italian art historian (see Vasari [1550, ed. 1568] cited in Feyerabend, 1999, pp. 89–90), who described how art progresses toward perfectly representing the real. Through stepping increments, saving the good from previous art and adding improvements, art reached perfection in Raphael of Urbino, whose "figures expressed perfectly the character of those they represented, the modest or the bold, being shown just as they are . . . , appear wholly realistic" (Feyerabend, 1999, pp. 89–90).

Mathematics proposes numbers to measure real things. There are notches corresponding to numbers on the measuring tape, but even if the notches succeed in referring to that real position. (although they remain a *sign* of the real object), gaps are still on the measuring tape with no notch and no number to describe the intermediate positions. The real number system attempts to fill the gaps that most numbers leave when describing something real, removing the need for metaphor. "Metaphorical language is language proper to the extent that it is related to the need for making up for gaps of language" (Giuliani, 1972, p. 131). The system "covers the gaps" and does the job of describing physical reality (and more) without metaphor. But how do real numbers go about covering the gaps?

AXIOM OF COMPLETENESS

The work of covering the gaps and freeing real numbers from metaphor is done with The Axiom of Completeness:

A bounded increasing sequence has a least upper bound (that is a real number)[2]

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[2] "A bounded monotone sequence has a limit" is the definition given by Mattuck (1999, p. 11). I replace monotone with increasing and limit with least upper bound for clarity.

Why would the axiom of completeness cover all the gaps of a real line? A good example is in the act of measuring a plank with a straight-looking side. One compares the plank with a measuring tape and measures the whole meters, but there is still some plank left to measure. (The number of whole meters is the first number (position) in the sequence.) So one counts the number of decimeters left (the resulting position is the second number in the sequence), but there still remains more plank after the largest marker for decimeters. The process continues until the precision of the measuring tape is exhausted, eyesight fails, or the measurer loses interest. Even though one must fail in measuring the exact length of the plank, the axiom of completeness provides assurances that there exists a real number for the "actual" length of the plank (and that there *is* an "actual" length of the plank).

But the process cannot take the full measure of the plank, and so we remain in the poetic world of metaphor, "a process, not a definitive act; it is an inquiry, a thinking on" (Hejinian, 2000). We want to talk about something real, something as simple and straightforward as the length of a plank. We have an apparatus of controlled inquiry, tools and will-more than the casual use of words, but we still fail. We must admit that the measurements (words) we have used remain metaphorical and the actual measure of the plank (object) ultimately falls into the gaps of language. The words (measurements) we started with in our task of measuring the plank are no less metaphorical than the measurement we have when we stop. How can we wake up from metaphor?

The axiom of completeness assumes that the lingering question: "What is the next level of precision for measuring this plank?" is immaterial because we could answer each question by continuing the measurement process forever, if we set ourselves to the task (perhaps we could always pass the task on to our children). More importantly it seems to assume that our words lose their metaphorical quality in the limit of the sequence, asserting realism: Berkeley, who would not accept there were things no-one perceived, said realism was ". . . the absolute existence of unthinking things without any relation to their being perceived . . ." (quoted in Borges, 1964, p. 173).

The measurement example gets at the notion that real numbers are real and out there, and our method for finding them involves

something like the scientific method. Charles Sanders Peirce, a philosopher traumatized with a rigorous mathematical education by his father, had this to say:

> It may be asked how I know that there are any Reals. If this hypothesis is the sole support of my method of inquiry, my method of inquiry must not be used to support my hypothesis. The reply is this: 1. If investigation cannot be regarded as proving that there are Real things, it at least does not lead to a contrary conclusion; but the method and the conception on which it is based remain ever in harmony. No doubts of the method, therefore, necessarily arise from its practice . . ." (Peirce, 1877/1992, p. 120)

The problem with Peirce's assertion is that there *are* doubts in any measurement of the plank. They take the form of questions. "Is that precise enough?" is a question that repeats at every measurement, and a measurement would not happen without the motivating question. What happens when the same doubt repeats over and over again? Often the question expands into other questions. "Why haven't I finished, am I doing something wrong?" To say no question of this kind *necessarily* arises is to put little faith in his students, as if they just do as their told.

The real number corresponding to the length of this plank is a least upper bound of a bounded (the plank is not infinite) sequence of measurements that is (the sequence) infinite and increasing. The measurement of the plank is an argument for the axiom of completeness, but it is incomplete because we cannot finish the measuring process, and we cannot speak about *all* such measurements (sequences mentioned in the Axiom). To more fully understand how the Axiom of Completeness creates the real numbers, and to discover the "escape" from metaphor, requires understanding how the Axiom is a "sweeping statement" (Borges, 1967).

INDUCTION

A generalization such as "All crows are black," made before we have complete knowledge of all crows, is the archetypal "sweeping statement." That move, called induction, leaps from the known—shaky as it is—to an unknown. The (often wrong) assumption is that

the unknown will be similar in some way to the known. The *Oxford English Dictionary* says induction is a logical movement from the specific to the general. In math, however, induction can go from the specific to the more specific. For example, take the decimal number derived from measuring. The numbers written down are not as specific as the finished decimal expansion, the real number, of the "actual" measure. The real number (represented with ". . ." for "the rest" because it cannot be fully written or expressed) refers to the findings of an infinite investigation. One cannot know the results and so uses induction to infer what the results might be.

In induction, confusion arises between the potential and the actual. Are we saying that the results might be or that they *are*? Aristotle mentions this confusion: ". . . to a never-ending process of division, we attribute an actuality which exists potentially." And this confusion is not surprising, because induction represents what we wish logic could do; what deduction cannot do.

Hume, the 18th century Scottish philosopher noticed what induction was based on: "that instances of which we have had no experience, must resemble those of which we have had experience, and that the course of nature continues always uniformly the same" (1888, p. 89). Concerning that uniformity he argues, "The principle cannot be proved deductively" (Vickers, 2012, p. 7). Any regularity in observed things could imply order or chaos in unobserved things, and neither is necessary deductively. Using induction would require we already knew the order of nature, which is circular. Hume reasoned that induction is not the work of reason, and so it must be the result of the imagination. "The force of induction, the force that drives the inference is thus not an objective feature of the world, but a subjective power, the mind's capacity to form inductive habits . . . inductive inference, is thus an illusion, an instance of what Hume calls the minds "great propensity to spread itself on external objects" (Hume, 1888, p. 167, cited in Vickers, 2012, p. 8).

Some texts do offer a proof of the axiom of completeness (Abbott, 2001, p. 245), but still use induction,[3] and "induction brings with it the risk of error" (Vickers, 2012, p. 5). Error can be found in the

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[3] Dedekind "cuts," defined as real numbers, are sets that can be characterized by a bounded, infinite and increasing sequence, just like the Axiom of Completeness. The difference is in name only. The infinite process is called an (open) set, and the set is then called a real number.

assumption that the unknown will resemble the known. But none of the possible error has a space in the "gapless" real number line. At each stage in the plank example, the measurement takes the form of action to find an answer that takes one back to a question—a dialogue between word and object, or between person and object. A dialectic model, raised before the real numbers arrive, does have room for the inductive error, as well as room for questions.

There is some history and difference of opinion on what induction is. Peirce says induction is something else, a logical move where the result is *different* from the premises (Peirce, "The Probability of Induction," 1878/1992, p. 162), anticipating Borges: ". . . the metaphor is a linking of two different things together . . ." (1967, Lecture, "The Metaphor"). The leap from metaphor to induction does not seem so wide under these definitions. But there is a difference. The real numbers emerged from many inductions, asserting a "natural" order to space and time. The belief in that nature shakes the sleeper awake, but the metaphor connecting waking life and the dream vanishes. The metaphor is replaced with induction, an unending task of refining words (measurements) toward actualizing the real numbers.

". . . that the rule of induction will hold good in the long run may be deduced from the principle that reality is only the object of the final opinion to which sufficient investigation would lead" (Peirce 1878, p. 169). When have we investigated the plank sufficiently? What about the rest of the plank? Where do we rest after the investigation, on the questions that result or on the final measurement? Peirce says in an earlier essay (1877) that uncertainty can only give dissatisfaction, and because of this "Reals" such as the "actual" measure of the plank cannot be doubted. Here he is ignoring skepticism, which practices a peaceful state in accepting a lack of certainty (see Pyrrhonism, in Suber, 1996).

The measurement of the plank is, metaphorically speaking, the progression of human knowledge and the movement from word to object. The axiom of completeness is the completion of this progression, this movement. We can begin to see why the Axiom is so useful. Once assumed as a premise, deduction[4] (like the usual Aristotelian logic) empowers us to demonstrate the synthetic quality

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[4] Deduction (like the usual Aristotelian logic) is not induction, it cannot tell anyone what they don't know based on what they know. It will only tell them what they already know in different ways.

of our knowledge, to find "new" knowledge. But, Peirce says, ". . . a synthetic inference cannot by any means be reduced to deduction . . ." (1878/1992, p. 162). Unfortunately his saying so cannot stop authors of scientific papers and teachers of mathematics, by assuming the Axiom of Completeness, from doing it anyway.

It seems we can imagine an awakening with the axiom of completeness. What is this awakening like? What is this new knowledge we have assumed?

WAKEFULNESS

In our example of measuring the plank we make a leap of logic to say something about *all* future measurements called induction. There is also an induction on all these inductions in the axiom of completeness, which talks not just about *this* plank and *this* induction on a sequence of measurements but about *all* of the similar types of measurements. The axiom of completeness ends up being the Induction of induction within the language of the real numbers. Just as the "King of kings" is another word for God, the Axiom of Completeness has ultimate say-so over the "real" world; every point in space is perfectly ordered and defined, it is imagined, well beyond the slightest detectible degree. An idealist would admit there is an infinite intelligence keeping an awareness of our ordered world everywhere we have not, or cannot look.

Mathematics is not alone in turning to God for answers on the non-metaphorical whatness of things. Aristotle, on the question "What is being?" said a composite thing was something in addition to its parts: ". . . this additional something is not an element but . . . the cause," and Aristotle's First Mover causes everything (*Metaphysics*, 1980, pp. 135–36).

The justification for the Axiom of Completeness is that without it numbers fail, just as words do, in describing the world. Successfully assuming the Axiom comes with a cost. The Axiom takes the students of real numbers down a path that has no possibilities, no holes or gaps where their decisions are important. Bugaev, a father figure in the rise of the mathematical school of Moscow, argued against the focus on completeness, saying incompleteness (or discontinuity) is a ". . . manifestation of independent individuality and autonomy" (Bugaev, 1897, quoted in Graham and Kantor, 2009, p. 68). For all the deductive thinking that follows in a math textbook on real numbers

(Abbott, 2001, Mattuck, 1999), it may be said that *thought itself has been excluded.* Any time that readers confront a situation of considering something, where they "really" may go one way or another with their thoughts, the book asserts necessity, and students don't have much choice but to follow along. The lingering questions are controlled: "What is the next level of precision?" and then all are answered, but only by saying the unfinished sequence of measurements "is finished" —the *number* —"is real."

This world of perfect wakefulness is also static. Zeno of Elea, a pre-Socratic philosopher and follower of Parmenides, believed all was one and thus change must be an illusion. Zeno argued that motion is not possible under the model of the real number line. One of his objections is illustrated in the story of Achilles and the Tortoise. The Tortoise is given a head start in his racing match against the demi-god. After the Tortoise runs a distance, Achilles begins running, and quickly runs the same distance. The tortoise has moved since Achilles started running, however, and is still ahead of Achilles by a shorter distance. Achilles runs this shorter distance in a small amount of time, but the Tortoise again has used the small amount of time to get ahead of Achilles. The result is an infinite sequence where the tortoise is always ahead of Achilles, and Achilles cannot win the race without completing an infinite task. Any motion of any speed, simply by hypothesizing a similar race, will require an infinite task to complete. The same argument can be used, not to prove motion is impossible, but that motion is not *expressible with points*. The problem Zeno put forth, the problem of completing an infinite task, is not dealt with in real number or calculus texts (Abbott, 2001), but is covered up under the name "limit." Having a name for an infinite task does not make it any less infinite. Aristotle writes that "every motion is incomplete" (*Metaphysics*, 1980, p. 152). In this perfect wakefulness, the light does not radiate, does not twinkle, no-one blinks.

The divine project of the language of real numbers, completed by the axiom, is reminiscent of the story of the Tower of Babel.

> In a mixture of positivism and myth, Dante attributes the rise of different languages to the occupational diversity required for building the Tower of Babel. The members of each trade or profession had their own language . . . . And perhaps with theological Latin in mind, he says that the higher the intellectual

quality of the specialty, the more barbarous the language. (Burke, 1950, p. 168)

Real numbers can't be written or spoken since their expression is infinite. On the language of the underworld in the Book of Thoth, "difficult are their words; their repetitions (or explanations) being too various to write . . ." (Jasnow, Lewis, and Zauzich, 2005, p. 63). We have conceived of an unspeakable language. In the silence, what has happened to poetry? The "thinking on" is over, now that we know everything. And poetry is when ". . . felt I like the watcher of skies / When a new planet swims into his ken" (Keats, 1816/1870).

The price of perfect wakefulness is paid by surrendering motion, and among the losses associated with that loss is the loss of discovery in poetry, and perhaps even life if Borges is right in saying, ". . . life is, I am sure, made of poetry" (Lecture, 1967). Imposing the Axiom of Completeness is not malicious, but springs from an intense desire to know. All is in motion, even things at rest move through time; *complete* certainty requires the end of motion. The Axiom stands for that surrender, that utter pessimism that knowing is better than being open to possibility. But the loss of motion is not well represented to math students, whose teachers instead claim that now they can fully describe—that is, *know*—motions with continuous functions (Abbott, 2001).

MOTION

Continuity has the connotation of movement. The real number line is continuous by the axiom of completeness. What is continuity? It is like "smoothness," but no one possesses a fine enough magnifying glass to ensure that a line remains uniform or telescope to ensure it remains smooth forever, and so we use induction to infer smoothness. Continuity of a line requires completeness or "gaplessness."

The "stock" metaphor for continuity of time is a river flowing (Borges, 1967). The prime quality of water is that no matter how you magnify, that is, divide out a smaller portion of it), the water remains the same. It is akin to saying water has no qualities, and closer inspection reveals the featurelessness of water. For the most part, water only takes on the features of its container. Although Heraclitus said, "you cannot go into the same water twice" (quoted by Plato, "The Cratylus," 1892, p. 92), the second time in, the water has all the

same features as the water the first time. Heraclitus also said, "We step and do not step in the same rivers, we are and are not" (Russell, 1918, p. 16). Continuity in math is like the experience of flowing water, and is also described intuitively as drawing the graph of a function without taking your pencil off the paper; the motion continues without sudden change or leap across a gap.

Newton asserted in his first law of motion: "Every body left to itself moves uniformly in a straight line" (quoted in Heidegger, 1977, p. 262). Zeno's objections aside, the real number line represents Newton's theory of motion best. The natural state of motion is a "uniform" or continuous one, and in a single direction. Heidegger writes that Newton begins in his first axiom of motion with "corpus omne," "every body," which means that the Aristotelian distinction between earthly and celestial bodies "has become obsolete" (Heidegger, 1977, p. 262). Aristotle asserted the existence of two "natural" motions of these two bodies: gravity and levity. Gravity moves in a straight line down, and levity moves up and eventually in circles in the heavens. But after Newton, the two natural motions no longer exist. Instead motion has only one nature—continuity or "uniformity" in a straight line, and one body described with real numbers.

The straight line requires opposition between positive and negative, or any other coupling of opposites. The spectrum of possible scores on the SAT, for example, places people along a line with two extremes and describes them with real numbers, and then uses the scores, monstrously, to decide in part a student's future. The "bodies" that live between any two opposites are of one type, described as real numbers. Their natural motion is of one type, in gradations that proceed continuously in a straight line. Time itself is no longer a "garden of forking paths" (Borges, 1941/1964), winding and separating, but one straight super highway to an infinite horizon. Escaping from the dream of metaphor involves following this highway, but succeeding the escape comes at the cost of accepting the Axiom.

The model of the straight line between opposites was not always universal. Heidegger offers that this model replaces any "agon" between opposites, where agon means strife or competition. Celestial motion was circular, which Aristotle argues cannot be defined in terms of an opposition (Aristotle, *Physics*, 1980, p. 176). One concept (such as the present you, reading) and the idea of constantly returning to the

present you, reading, is enough for circular motion. What motivation could there be for Newton, a devout Christian with a famous saintly persona, to minister the world out of the heavenly movements, into a world characterized by competition of opposites, and survival of the fittest?

In our example of measuring the plank, the dialectic process is what some call the essence of motion. Apart from Newton, Marx believed the human world would eventually be characterized by free association and cooperation (Marx, 1875). Lenin reported that Marx believed dialectics is "the science of the general laws of motion both of the external world and of human thought" (Lenin, 1980). Teachers of analysis encourage thinking through a proof of continuity with the dialectic relationship between two shrinking quantities called *epsilon* and *delta.* This is because *epsilon* is a response to an arbitrary choice of the value (or dialectical position) of *delta.* After thinking it through dialectically the student removes the dialectic "shape" and replaces it with a logical proof.

The quantities *epsilon* and *delta* are "open sets." The definition of a real open set "G" requires a notion of a "neighborhood" around every point "a" in "G" that is complete like the real numbers. Such a *positive* definition of an open set is possible because of the Axiom of Completeness, because the neighborhood is complete. Without the axiom of completeness, open sets would have to be *dialectically* defined. "[T]he dialectical definition does not aim in short at determining the essence, but the limits of a proposed definition" (Guilani, 1972, p. 131). Open sets are simply not their boundaries. When you say "not the boundary (or limit)" you indirectly get the same completeness of an open set, but without a positive definition. Some boundaries would require on-going inquiry to find. Dialectically defined open sets make for a different kind of mathematics of continuity, difficult in different ways, but the basic idea of an open set is easier to state and oppose to "closed sets" (the opposite of an open set, defined as a set that contains all its "boundaries" or limit points).

Continuity (and movement) need not be a result of the Axiom of Completeness, because a dialectic process can describe continuity instead (incompletely, vaguely). The axiom of completeness removes the need for dialectically defined objects, as well as metaphor, within the language of the real numbers. Life goes on, but after the acceptance of the Axiom life has changed. Followers of the Axiom

have no place in time or space. "In time, because if the future and the past are infinite, there will not really be a when; in space, because if every being is equidistant from the infinite and the infinitesimal, there will not be a where. No one exists on a certain day, in a certain place; no one knows the size of his face." (Borges, 1964, p. 8)

The wakefulness we find from the Axiom seems to falter, and thinking about what should be done continues. Questions motivate and connect the various positions in dialectical process. Without questions there can be no internal motion, but Aristotle would extend this to the external world by saying motion cannot exist without potential. Questions are an expression of many potential answers, many possible paths. Do we follow this image of the real line, a single super highway to the horizon, or do we wonder which path takes us there?

QUESTIONS

Aristotle believes levity is a natural motion upwards possessed by only certain kinds of bodies. Today Newton's laws of motion include gravity but not levity and only one kind of body, all under the pull of gravity. Questions, at least internally, offer a necessary form of thought, different from answers such as real numbers, a form of thought that expands, and leads to an uplifting sense of wonderment. Levity has not left us internally, but what about externally? Is levity real, with all the "weight" that reality must have?

What are questions? For an example of the question itself, "This is a question" doesn't work because it is not a question. Referring to the question with a question: "Is this a question?" seems to be an example of the question itself. Is it a question whether we ask it or not, the way answers are discovered? If the answer is no, then "Is this a question?" isn't always a question. That is, if we write it down in a book and forget about it, that it is a question ceases to be true. As soon as we stop asking it, it would have to disappear. If I confirm "Is this a question" is a question over and over again, using induction is appropriate to say that "Is this a question?" is, internally, a real question. But, the use of induction here brings us to a paradox- as soon as we use induction to conclude that "Is this a question?" IS a question, "Is this a question?" is not in question. After all, how could this question be, if its being were not in question? We cannot even say that "Is this a question?" is real *internally*, but there is still the problem that every time I ask, and before I answer, "Is this a question?" it

seems to be a question. So we rest, for now, on the conclusion that the question is, internally, not unreal. We are limited on both sides, inconclusively, the inductive conclusion that the question is real cannot be made, yet the question itself is confirmable in a way that makes it seem real.

Questions are elusive and hard to pin down as objects of thought, yet they are indispensable for dialectical process. Thought itself moves naturally between certainty and inquiry. We use questions every day, and looking for a definition of the question justifies the dialectical definition in general; even though it is imprecise it is the best we can do. It is expected that questions will not be easily discovered externally, as potential, but will naturally avoid definition. Questioning is a necessary part of dialectical movement, therefore questions are a necessary part of thought. If thought has any physical reality (perhaps in the brain), then so does questioning have a physical reality. It has been shown that it is not so simple to claim that the question is "real." Perhaps exploring the question as an expression of potential can help.

A rock on the top of a hill has potential energy. The potential to move a long way by rolling down the hill, but where, on what plane, is this potential perceivable? The rock has potential because it has *not* rolled down the hill. An empty vessel has the potential to be filled, as does a gap in language. Metaphor fills gaps by linking the gap to other words. The link is like a part of a web. Because of our difficulty with defining the question itself, it is understandable to be wary of settling on the idea that potential is "not-being" or a hole or gap. The problem of defining power is more difficult than "not-being"; the lack of a person in front of me does not mean there is the potential for one to appear magically before me. The web is an apt metaphor for power- it must have holes to be, but not everything is possible because the web has a structure in its links. Foucault argued the model for power was a web, but well before that the Greek myth of Arachne made it clear the webs humans weave have the power to rival the gods. In the Book of Thoth it is written, "What is the taste of the prescription of writing? What is this net?" (Jasnow, Lewis, and Zauzich, 2005, p. 16)

Numbers *before* the Axiom of Completeness are arranged in a web, like the grid on graph paper, but the web of lines on the graph paper is Euclidean, and that is a particular kind of web with parallel lines that are straight and equidistant. The Euclidean web makes

distance measurable with numbers. There are other kinds of webs. A web with curved links is not measureable unless it is translatable into a Euclidean web.[5]

A common argument for real numbers filling the gaps of a line is to draw a square with a side equal to one unit and then draw the diagonal inside the square. The diagonal, by the Pythagorean theorem, has a length of the square-root of two, which turns out to be a hole in the smaller number systems before the real numbers. The argument is to show someone a length with no number to emphasize the need for a better number system. Such an argument depends on Euclidean space, however, and depends on the perfectly straight lines found in Euclidean space. The epic history of mathematicians trying to prove Euclidean geometry as the one true geometry has never ended in success (Hartshorne, 2000, p. 304). Instead, other geometries have been admitted for study, but where does that leave our argument for the existence of the square-root of two? The construction of the square and its diagonal is just one possibility, one rhetorical posture. Possibilities for rhetorical argument *against* the existence of real numbers are also available using different "webs."

Thinking in a language with *no* gaps such as the real numbers, is thought with *no* potential. The real numbers do not make a web; they are "completely" connected. The real numbers create a "real" world of orderly and knowable space and time, and in doing so it has a power to completely describe this "real" world, so that real numbers seem to be "all powerful" (Hartshorne, 2000, p. 3). The perfectly known world lacks potential "completely," however. Numbers are powerful, but the rhetorical choice of words such as "real numbers" (potential numbers?) and "completeness" is a treatment of infinity as an actuality, and make it clear that the power of numbers is not to be given to math students. The secret to the real numbers is there *is* a hole- and that hole is the lack of potential, of "The Question."

Questions are not unreal. They are not included in Newton's statement about "every body," since they cannot be represented with real numbers (which are answers). Mathematicians attempt to know "real" numbers even though there are more of them than can be literally expressed. Questions are capable of suggesting more than

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[5] Unless we grossly redefine distance, and in these cases there are multiple viable definitions for distance. I do not use "translate" in a technical sense.

they are capable of literally expressing- they suggest many answers at once, in a more simple way than real numbers do. Taking the inductive leap to say questions are real only arrives at a paradox. We cannot say questions, in general, are real. Is the paradox a contradiction? Is "Is this a question?" a material counter-example against induction? The spaces that these questions inhabit are not to be found on the real line; numbers cannot answer them.

CONCLUSION

> Concerning the basic law of motion, the law of inertia, the question arises whether this law is not to be subordinated under a more general one, i.e., the law of conservation of energy, which is now determined according to its expenditure and consumption, as work, . . . questionability is concealed by the results and the progress of scientific work." (Heidegger, 1977, p. 270)

Statistics, physics, and economics do work that depends on the Axiom of Completeness, as does any topic that uses continuous functions. High school mathematics textbooks assume the axiom of completeness implicitly, and the Axiom enters explicit study up to the graduate level. The work that college math students of the Axiom then do tends to proceed deductively, and so can be clear only internally (the internal logic is clearly necessary), but the meaning of their work—the external synthesis with the world- their Metaphor—is not explored. It is certain that such a synthesis is of epic proportion, but with only deductions, the mathematics amounts to externally mysterious re-phrasings of the Axiom. Math textbooks (e.g., Mattuck, 1999) use extremely short summaries of the deep philosophical and rhetorical problems of the Axiom and offer them as assumptions (Abbott, 2001), then conceal the questionability of the assumptions by directing students to work at solving problems and deducing theorems. Having students accept what the textbooks say without discussion, controversy and reading into the deep problems of completeness, continuity, and motion is the opposite of education.

Aristotle would object that there are no holes or gaps found anywhere (Aristotle, *Physics,* 1980), and perhaps in space, motion cannot be in a vacuum, but the old theory of ether resurfaces to explain movement: "the ether hypothesis was bound always to play

some part in physical science, even if at first only a latent part" (Einstein, 1922). And this latent part it plays is as a medium. "Newtonian . . . [gravity] . . . is only apparently immediate action at a distance, but in truth is conveyed by a medium permeating space, whether by movements or by elastic deformation of this medium." (Einstein, 1922). If so this medium could not be thought of as abundant in spaces and rare in particles, which would require another medium between these particles. Instead the metaphor for "ether" could be as a "cloud," and not the way a cloud in the sky is thought of today which is as particles and space. Instead, if one must think in particles/points, it may be thought that the particles are in more than one place, or in indefinite spaces at once. Unlike the Axiom of Completeness, which asserts every position is knowable, these "clouds" are vague notions representing potential instead of certainty; they are bodies of levity. They are not unreal, but they are not exactly there; they are here as well.

    The proof that the square root of two is not a rational number (and therefore is one of the holes the Axiom fills with real numbers) is essentially a poem, grasping at the inexpressible and failing. The square-root of 2 is indefinite, and the statement, "The square-root of 2 is a quantity," are spoken metaphorically.

    "The Question" overarches and connects difficult questions in philosophy, such as the role of potential in the hard problem of motion, what is knowable, or whether humans are dreams or real. The axiom of completeness defines the numbers we use to describe ourselves, our futures (GPAs, exam scores). The Axiom seems to be "The Answer" in a state of opposition to "The Question." The Axiom is so extreme that arguing against it is polarizing. The reality is that there is not one path away from the Axiom. Any thought or decision multiplies these forking paths, making us wonder more and more. Motion and its dialogue continue. "Have I dreamt my life or was it a true one?" The German poet Vogelweide asked in his poem, "Ah! Where are the hours departed fled!" (c. 1200). Comparing Vogelweide's line to Shakespeare's "We are such stuff as dreams are made on," Borges offers, "instead of a sweeping affirmation, we have a question . . . and this hesitation gives us that dreamlike essence" (Borges, "The Metaphor," 1967).

    Ah! where are hours departed fled?

Is life a dream, or true indeed?
Did all my heart hath fashioned
　　From fancy's visitings proceed?
Yes! I have slept; and now unknown
　　To me the things best known before:
The land, the people, once mine own,
　　Where are they? — they are here no more . . .
　　　　　　　　　　　　—Vogelweide, 1750–1220

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If you have any questions, concerns or comments about this essay you are welcome to email me at:

Zweihanderdawg@gmail.com.