A note to the reader. This is the most technically demanding piece in the book, and it is addressed to readers with some background in formal logic or mathematics. If you have never encountered truth tables, proof systems, or formal semantics, you are very welcome to skip directly to the next section; nothing in the rest of the book depends on what follows. What this paper does is put precise formal machinery around a philosophical idea that is argued everywhere else in non-technical terms: that holding a question open is not a failure of thought but a legitimate logical status. For those who speak the language of logic, the machine below is the same argument in a different dress. For everyone else, the dress is optional.

$L_?$: A Questioning Logic

Inspired by Vagueness as the Embodiment of Inquiry

Andrew Nightingale · November 29, 2025

Abstract

This paper introduces a propositional logic, $L_?$, that formalizes the idea that some sentences are best treated as held open rather than assigned a classical truth value. The system is inspired by a view of vagueness as the embodiment of inquiry: vagueness marks the boundary where language meets the world and questions open up, rather than a mere defect to be eliminated. Technically, $L_?$ enriches a Strong Kleene three-valued base with a unary "questioning" operator ? and supports two consequence relations: a strict consequence relation $\vdash_s$ and an inquiry-oriented relation $\vdash_q$. We establish the structural properties of both relations, identify key validities and invalidities, relate $L_?$ to the familiar logics K3 and LP, sketch a Hilbert-style proof system, prove soundness, and note decidability. A single-relation variant $L_?^*$ is also presented. The main virtue of the system is conceptual clarity: it gives a formal home to the act of questioning without reducing questions to ignorance, error, or meaninglessness.

Contents

1 Motivation

On the view motivating $L_?$, vagueness is not merely a gap in knowledge or a failure of precision; it is the embodiment of inquiry. We encounter vague cases whenever our existing concepts and linguistic forms are strained by the world, and the natural response is not always to force a sharp classification, but sometimes to leave a statement open—to keep it in play as a question.

Traditional responses to vagueness and paradox often proceed by either:

- assigning non-classical truth values while preserving classical-style consequence (e.g. Strong Kleene logic, K3), or
- declaring certain sentences simply "meaningless" and excluding them from the logic (Bochvar's external approach).

The proposal here is different in spirit. We introduce an explicit connective, ?, that marks a sentence as being held in a state of inquiry. Formulas of the form $?\varphi$ are not treated as settled truths, but they are not discarded either. They live in a third status: open, but admitted into discourse.

Technically, we work over a Strong Kleene three-valued base, with truth values $\{1, u, 0\}$ corresponding to true, open, and false. The operator ? sends any formula to the open value u. We then distinguish:

- a strict consequence relation $\vdash_s$, which cares only about settled truth (1), and
- an inquiry consequence relation $\vdash_q$, which treats both 1 and u as acceptable outcomes.

This dual structure lets us formalize the act of taking problematic or vague sentences out of the "strict" layer of reasoning while keeping them present as live questions.

Why not simply use K3 or LP? Strong Kleene logic (K3) uses the same three values but designates only 1: the indeterminate value u is, in K3, a kind of semantic quarantine—a value that neither transmits to conclusions nor contributes positively to discourse. Priest's Logic of Paradox (LP) goes the other direction, designating both 1 and u throughout. $L_?$ contains both as special cases (see Section 9) but adds the ? operator, which lets us explicitly mark which formulas we are holding open—a distinction that neither K3 nor LP can draw.

2 Language and Semantics of $L_?$

2.1 Syntax

The language $L_?$ is defined as follows:

- *Propositional variables:* $p, q, r, \ldots$
- *Connectives:* $\neg$, $\wedge$, $\vee$, $\rightarrow$

- *Unary questioning operator:* ?

Formulas are built in the usual way. If φ is a formula, then $?\varphi$ is a formula, read as:

$$?\varphi \;=\; \text{“we hold } \varphi \text{ open; } \varphi \text{ is under inquiry.”}$$

2.2 Truth Values

We use three truth values:

$$\{1,\ u,\ 0\}$$

with the intended reading: $1 =$ true; $u =$ open / under inquiry; $0 =$ false.

A *valuation* v is a map from propositional variables to $\{1, u, 0\}$, extended to all formulas via the clauses below.

2.3 Base Connectives: Strong Kleene

For the standard connectives, we adopt Strong Kleene tables.

Negation.

$$v(\neg\varphi) \;=\; \begin{cases} 0 & \text{if } v(\varphi) = 1, \\ 1 & \text{if } v(\varphi) = 0, \\ u & \text{if } v(\varphi) = u. \end{cases}$$

Conjunction. We treat $\wedge$ as a minimum operation on the order $0 < u < 1$:

$$v(\varphi \wedge \psi) = \min\bigl(v(\varphi),\ v(\psi)\bigr).$$

$\wedge$	1	u	0
1	1	u	0
u	u	u	0
0	0	0	0

Disjunction. We treat $\vee$ as a maximum operation on $0 < u < 1$:

$$v(\varphi \vee \psi) = \max\bigl(v(\varphi),\ v(\psi)\bigr).$$

$\vee$	1	u	0
1	1	1	1
u	1	u	u
0	1	u	0

Conditional. We define the conditional via $\varphi \to \psi := \neg\varphi \vee \psi$, using the above clauses.

$\to$	1	u	0
1	1	u	0
u	1	u	u
0	1	1	1

2.4 The Questioning Operator

Definition 2.1 (Semantics of ?). For any valuation v and any formula φ,

$$v(?\varphi) \;:=\; u.$$

Intuitively, $?\varphi$ explicitly forces the "open" value. Whatever the underlying status of φ may be, the act of asserting $?\varphi$ is the act of refusing to treat φ as settled true. Formulas of the form $?\varphi$ are therefore never evaluated as 1 or 0; they are always u.

Remark 2.2. This makes ? a constant-value operator: its output is independent of its input's semantic value. This is a deliberate design choice. A more refined operator might interact with the underlying value of φ (e.g., $?(?\varphi) = u$ but $?(\varphi \wedge \neg\varphi)$ might behave differently). The constant design keeps the present system maximally simple while still capturing the key idea: the act of questioning is not a claim about the subject matter but a stance toward it.

3 Two Consequence Relations

Definition 3.1 (Strict Consequence). Let Γ be a set of formulas and φ a formula. We say that φ is a *strict consequence* of Γ, written $\Gamma \vdash_s \varphi$, iff for every valuation v: if $v(\gamma) = 1$ for all $\gamma \in \Gamma$, then $v(\varphi) = 1$.

Here only the value 1 is designated.

Definition 3.2 (Inquiry Consequence). Let Γ be a set of formulas and φ a formula. We say that φ is an *inquiry consequence* of Γ, written $\Gamma \vdash_q \varphi$, iff for every valuation v: if $v(\gamma) = 1$ for all $\gamma \in \Gamma$, then $v(\varphi) \in \{1, u\}$.

For $\vdash_q$, the designated values are $\{1, u\}$.

Inquiry consequence says: when the premises are settled true, the conclusion is at least not false. We allow the conclusion to be genuinely open while counting it as acceptable within an inquiry.

Remark 3.3. Since $v(?\varphi) = u$ for all v, the formula $?\varphi$ is always $\vdash_q$-designated (because $u \in \{1, u\}$) but never $\vdash_s$-designated (because $u \neq 1$). Questioned formulas participate in inquiry without counting as strict theorems.

Proposition 3.4 (Monotone inclusion). $\vdash_s \subseteq \vdash_q$. *That is, if* $\Gamma \vdash_s \varphi$ *then* $\Gamma \vdash_q \varphi$.

Proof. Suppose $\Gamma \vdash_s \varphi$. Let v be any valuation with $v(\gamma) = 1$ for all $\gamma \in \Gamma$. Then $v(\varphi) = 1$, hence $v(\varphi) \in \{1, u\}$. Since v was arbitrary, $\Gamma \vdash_q \varphi$. □

Proposition 3.5 (Strict inclusion). *The inclusion is strict: there exist* Γ, φ *such that* $\Gamma \vdash_q \varphi$ *but* $\Gamma \nvdash_s \varphi$.

Proof. Take $\Gamma = \emptyset$ and $\varphi = p \vee \neg p$. For any valuation v: if $v(p) = 1$, then $v(p \vee \neg p) = \max(1, 0) = 1$; if $v(p) = 0$, then $v(p \vee \neg p) = \max(0, 1) = 1$; if $v(p) = u$, then $v(p \vee \neg p) = \max(u, u) = u$. So $v(p \vee \neg p) \in \{1, u\}$ for all v, giving $\vdash_q p \vee \neg p$. But $v(p \vee \neg p) = u$ when $v(p) = u$, so $\nvdash_s p \vee \neg p$. □

4 Structural Properties

We establish that both relations behave well structurally.

Proposition 4.1 (Reflexivity)**.** *For any formula φ: $\{\varphi\} \vdash_s \varphi$ and $\{\varphi\} \vdash_q \varphi$.*

Proof. Immediate from the definitions. □

Proposition 4.2 (Monotonicity)**.** *Both relations are monotone: if $\Gamma \vdash_s \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_s \varphi$; and likewise for $\vdash_q$.*

Proof. If $v(\gamma) = 1$ for all $\gamma \in \Delta$, then certainly $v(\gamma) = 1$ for all $\gamma \in \Gamma$, so the conclusion follows from the hypothesis. □

Proposition 4.3 (Cut)**.** *Both relations satisfy Cut: if $\Gamma \vdash \varphi$ and $\Delta, \varphi \vdash \psi$, then $\Gamma, \Delta \vdash \psi$ (for either $\vdash$).*

Proof. We show the case for $\vdash_s$; the case for $\vdash_q$ is analogous. Let v make all of $\Gamma \cup \Delta$ true at value 1. By the first hypothesis $v(\varphi) = 1$. By the second hypothesis, since φ and all of Δ are valued at 1, we get $v(\psi) = 1$. □

Remark 4.4. These three properties—Reflexivity, Monotonicity, Cut—characterize the structural skeleton of a Tarskian consequence relation. Both $\vdash_s$ and $\vdash_q$ are therefore well-behaved in this sense, even though they diverge in what they designate.

5 Key Validities and Invalidities

We write $\vdash \varphi$ (with empty premise set) for a tautology.

Proposition 5.1 (No strict tautologies)**.** *$L_?$ has no strict tautologies: $\nvdash_s \varphi$ for any formula φ built solely from propositional variables and standard connectives (without $?$).*

Proof. Assign $v(p) = u$ for every propositional variable p. One verifies by induction that $v(\varphi) = u$ for every ?-free formula φ: the Strong Kleene connectives all map (u, u) to u. Hence $v(\varphi) \neq 1$, so φ is not a strict tautology. □

Proposition 5.2 (Inquiry tautologies)**.** *The following are $\vdash_q$-tautologies:*

(i) $p \vee \neg p$ *(Law of Excluded Middle)*
(ii) $\neg(p \wedge \neg p)$ *(Law of Non-Contradiction)*
(iii) $?\varphi$ *for any formula* φ *(All questions are inquiry-valid)*
(iv) $p \to p$

Proof. (i) As shown in Proposition 3.5. (ii) $v(\neg(p \wedge \neg p))$: if $v(p) = 1$, then $v(p \wedge \neg p) = \min(1, 0) = 0$, $v(\neg(\cdot)) = 1$; if $v(p) = 0$, then $v(p \wedge \neg p) = \min(0, 1) = 0$, $v(\neg(\cdot)) = 1$; if $v(p) = u$, then $v(p \wedge \neg p) = u$, $v(\neg(\cdot)) = u$. All values in $\{1, u\}$. (iii) $v(?\varphi) = u \in \{1, u\}$ always, by Definition 2.1. (iv) $v(p \to p) = v(\neg p \vee p) = \max(v(\neg p), v(p))$, which equals 1 when $v(p) \in \{1, 0\}$ and u when $v(p) = u$; always in $\{1, u\}$. □

Proposition 5.3 (Behavior of classical principles under $\vdash_s$). *(i)* $\nvdash_s p \vee \neg p$ *: Excluded Middle fails as a strict tautology. When* $v(p) = u$*, we get* $v(p \vee \neg p) = u \neq 1$.

(ii) $p \wedge \neg p \vdash_s q$ *for any formula* q *: since* $v(p \wedge \neg p) \in \{0, u\}$ *for all* v*, the premise set* $\{p \wedge \neg p\}$ *has no valuation assigning value* 1*, so the inference holds vacuously. See Remark 5.4.*

(iii) $p \to q,\ p \vdash_s q$ *(Modus Ponens holds) : see Proposition 5.5.*

Remark 5.4 (Vacuous ex contradictione). In $\vdash_s$, note that $v(\varphi \wedge \neg\varphi) \in \{0, u\}$ for all v (it can never be 1). Therefore the premise set $\{\varphi \wedge \neg\varphi\}$ has no valuation that makes the premise equal 1, so $\varphi \wedge \neg\varphi \vdash_s \psi$ holds vacuously for every ψ. This is a classical feature. What fails is the non-vacuous ex falso: if $v(\varphi) = u$, the formula $\varphi \wedge \neg\varphi$ has value u, not 1, so a contradiction in the open sense does not license strict inference. The system thus blocks the most dangerous form of explosion at the inquiry level.

Proposition 5.5 (Modus Ponens holds for $\vdash_s$). $p,\ p \to q\ \vdash_s\ q$.

Proof. Suppose $v(p) = 1$ and $v(p \to q) = v(\neg p \vee q) = 1$. Since $v(p) = 1$, $v(\neg p) = 0$. Then $\max(0, v(q)) = v(q) = 1$. So $v(q) = 1$. □

Proposition 5.6 (Questioned formulas cannot serve as live premises). *Since* $v(?\varphi) = u \neq 1$ *for all valuations* v*, no valuation satisfies the premise condition of Definitions 3.1 or 3.2 with* $?\varphi$ *as a premise. As a consequence:*

(i) $?\varphi \vdash_s \psi$ *for every* ψ*, vacuously: no valuation makes* $?\varphi$ *equal to* 1*, so the antecedent of Definition 3.1 is never triggered. This is the exact parallel of the vacuous* $\varphi \wedge \neg\varphi \vdash_s \psi$ *noted in Remark 5.4.*

(ii) Similarly, $?\varphi \vdash_q \psi$ *for every* ψ *that is a* $\vdash_q$*-tautology, and also holds vacuously for all remaining* ψ*, for the same reason.*

(iii) The blocking of these vacuous inferences in practice is achieved proof-theoretically: the blocking rule of Section 6 prohibits $?\varphi$ *from serving as a premise in Modus Ponens steps. The semantics and the proof system thus come apart by design for questioned formulas: the inferences are semantically valid (vacuously) but proof-theoretically blocked.*

The philosophical point stands: questioned formulas do not function as live inferential premises. The operational locus of that point is the proof system, not the semantics.

6 A Hilbert-Style Proof System for $\vdash_s$

We sketch a Hilbert-style axiomatization of $\vdash_s$ for the ?-free fragment, extending to ?-containing formulas via blocking rules.

6.1 Axiom Schemas

The following schemas are valid under $\vdash_s$:

Propositional Core (K3-valid schemas):

$$\varphi \to (\psi \to \varphi) \tag{K}$$
$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \tag{S}$$
$$(\neg\neg\varphi \to \varphi) \land (\varphi \to \neg\neg\varphi) \tag{DN}$$
$$\varphi \to (\varphi \lor \psi) \tag{$\lor$I}$$
$$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \lor \psi) \to \chi)) \tag{$\lor$E}$$
$$(\varphi \land \psi) \to \varphi \text{ and } (\varphi \land \psi) \to \psi \tag{$\land$E}$$
$$\varphi \to (\psi \to (\varphi \land \psi)) \tag{$\land$I}$$
$$(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi) \tag{Contra}$$

*Schemas that are **not** axioms (classically valid but not K3-valid under $\vdash_s$):*

- $\varphi \lor \neg\varphi$ (Excluded Middle — fails when $v(\varphi) = u$)
- $(\varphi \to \psi) \lor (\psi \to \varphi)$ (Linearity — fails)
- $((\varphi \to \psi) \to \varphi) \to \varphi$ (Peirce's Law — fails)

6.2 Rules

$$\frac{\Gamma \vdash_s \varphi \qquad \Gamma \vdash_s \varphi \to \psi}{\Gamma \vdash_s \psi} \text{ (Modus Ponens)}$$

Blocking rule for ?: Formulas of the form $?\varphi$ are not permitted as premises in any Modus Ponens step within $\vdash_s$-derivations. They may appear only as conclusions. This rule intentionally prevents derivation of vacuously valid inferences of the form $?\varphi \vdash_s \psi$ (see Proposition 5.6): such inferences are semantically valid but proof-theoretically blocked by design.

6.3 Rules for $\vdash_q$

The inquiry consequence relation adds:

$$\frac{\Gamma \vdash_s \varphi}{\Gamma \vdash_q \varphi} \text{ (Strict-to-Inquiry)} \qquad \frac{}{\Gamma \vdash_q ?\varphi} \text{ (Question-Introduction, any } \Gamma)$$

$$\frac{\Gamma \vdash_q \varphi \qquad \Gamma \vdash_q \psi}{\Gamma \vdash_q \varphi \land \psi} \text{ ($\land$-Inquiry)}$$

Theorem 6.1 (Soundness)**.** *If φ is derivable from Γ in the above system for $\vdash_s$ (resp. $\vdash_q$), then $\Gamma \vdash_s \varphi$ (resp. $\Gamma \vdash_q \varphi$) in the sense of Definitions 3.1 and 3.2.*

Proof. By induction on the length of derivation. Each axiom schema is verified to take values in $\{1\}$ (for $\vdash_s$) or $\{1, u\}$ (for $\vdash_q$) under all valuations that make the premises hold. Modus Ponens preserves value 1 as shown in Proposition 5.5. The Question-Introduction rule is sound for $\vdash_q$ because $v(?\varphi) = u \in \{1, u\}$ always. The blocking rule ensures that $?\varphi$ never appears as a strict premise, preventing its vacuously valid strict consequences from entering derivations. □

Remark 6.2 (On completeness). A completeness theorem for $\vdash_s$ restricted to ?-free formulas follows from the known completeness of K3 consequence (Kleene 1952; Urquhart 2001). Extending completeness to the full language with ? is straightforward: $v(?\varphi) = u$ is a complete semantic characterization of the questioning operator, and the Question-Introduction rule captures it exactly. A detailed completeness argument is left as a routine exercise for readers familiar with the Lindenbaum-Tarski method.

Note that the Hilbert system is *intentionally incomplete* for ?-containing formulas in one respect: the blocking rule prevents derivation of $?\varphi \vdash_s \psi$ even though this inference holds vacuously at the semantic level. This is a deliberate design choice: the proof system is meant to track live, non-vacuous inferential connections, not to enumerate all semantic validities.

Proposition 6.3 (Decidability). *Both $\vdash_s$ and $\vdash_q$ are decidable.*

Proof. The truth values are finite ($\{1, u, 0\}$), and the connectives—including ?—are computable. For any finite premise set Γ and conclusion φ, one can enumerate all 3^n valuations (where n is the number of propositional variables in $\Gamma \cup \{\varphi\}$) and check the condition in Definitions 3.1 or 3.2 in finite time. □

7 Rules Governing the Questioning Operator

7.1 Question-Introduction

Definition 7.1 (Question-Introduction (informal)). From any formula φ, it is permissible to introduce $?\varphi$.

Proposition 7.2 (Contextual Question-Introduction). *If $\Gamma \vdash_s \varphi$, then $\Gamma \vdash_q ?\varphi$.*

Proof. Suppose $\Gamma \vdash_s \varphi$. Let v make all of Γ true. Then $v(\varphi) = 1$, but $v(?\varphi) = u \in \{1, u\}$. Hence $\Gamma \vdash_q ?\varphi$. □

This captures the idea that we may re-open a sentence even when it has previously been treated as strictly true. A settled matter can always be placed back under inquiry.

7.2 Question-Elimination is not Possible in General

Proposition 7.3 (No question-elimination under $\vdash_q$). *For any non-tautological formula φ:*

$$?\varphi \nvdash_q \varphi.$$

Under $\vdash_s$, $?\varphi \vdash_s \varphi$ holds vacuously, as a special case of Proposition 5.6 (i).

Proof. For $\nvdash_q$: since φ is not a $\vdash_q$-tautology, there exists a valuation v with $v(\varphi) = 0$. That same valuation has $v(?\varphi) = u \in \{1, u\}$ (so the $\vdash_q$-premise condition is satisfied) but $v(\varphi) = 0 \notin \{1, u\}$. Hence $?\varphi \nvdash_q \varphi$.

The vacuous $\vdash_s$ case follows from the general argument of Proposition 5.6 (i): since no valuation makes $?\varphi = 1$, the $\vdash_s$-antecedent is never triggered. □

Remark 7.4. The $\vdash_q$ direction carries the philosophical weight: holding a question open does not commit us to any answer. Questions are epistemically neutral with respect to their conclusions. The $\vdash_s$ direction is a vacuous artifact of unsatisfiable premises, not a meaningful inference; its force is entirely absorbed by the proof-theoretic blocking rule of Section 6.

7.3 Idempotence of ?

Proposition 7.5 (? is idempotent)**.** *$v(?\,?\varphi) = v(?\varphi) = u$ for all v. Hence $??\varphi$ and $?\varphi$ are semantically equivalent.*

Proof. By Definition 2.1, $v(?\varphi) = u$ for any φ. Applying the definition again, $v(??\varphi) = v(?\,(?\varphi)) = u$. □

Questioning a question does not deepen or alter the inquiry status—a pleasing result. Iterating ? produces no new information.

7.4 Interaction of ? with Connectives

The following identities capture how ? distributes over (or fails to distribute over) standard connectives.

Proposition 7.6 (Semantic identities involving ?)**.** *For all valuations v:*

(i) $v(?\varphi \wedge ?\psi) = u$
(ii) $v(?\varphi \vee ?\psi) = u$
(iii) $v(\neg ?\varphi) = u$ *(The negation of a question is also open.)*
(iv) $v(?\varphi \to \psi) = \max(u,\, v(\psi))$*, which equals* 1 *if* $v(\psi) = 1$*, and* u *if* $v(\psi) \in \{u, 0\}$*.*
(v) $v(\varphi \to ?\psi) = \max(v(\neg\varphi),\, u)$*, which equals* u *if* $v(\varphi) \in \{1, u\}$*, and* 1 *if* $v(\varphi) = 0$*.*

Proof. Direct calculation using Definition 2.1 and the Kleene tables.

For (iv): $v(?\varphi \to \psi) = \max(v(\neg\,?\varphi),\, v(\psi)) = \max(u,\, v(\psi))$, since $v(\neg\,?\varphi) = u$. Cases: $v(\psi) = 1$: $\max(u, 1) = 1$; $v(\psi) = u$: $\max(u, u) = u$; $v(\psi) = 0$: $\max(u, 0) = u$.

For (v): $v(\varphi \to ?\psi) = \max(v(\neg\varphi),\, v(?\psi)) = \max(v(\neg\varphi),\, u)$. Cases: $v(\varphi) = 1$: $v(\neg\varphi) = 0$, $\max(0, u) = u$; $v(\varphi) = u$: $v(\neg\varphi) = u$, $\max(u, u) = u$; $v(\varphi) = 0$: $v(\neg\varphi) = 1$, $\max(1, u) = 1$. □

Remark 7.7. Item (iii) is philosophically significant: $\neg ?\varphi$ has value u, not 1 or 0. The negation of a question is itself open—it does not close the question by asserting the negative. This blocks a common rhetorical move: one cannot simply negate a held-open sentence to produce a settled falsehood.

Items (iv) and (v) each depend on the three-case analysis rather than a two-case shortcut. For (iv): the formula $?\varphi \to \psi$ takes value u not only when $v(\psi) = 0$ but also when $v(\psi) = u$; it takes value 1 only when $v(\psi) = 1$. For (v): the formula $\varphi \to ?\psi$ takes value u not only when $v(\varphi) = 1$ but also when $v(\varphi) = u$, since $\max(u, u) = u$; it takes value 1 only when $v(\varphi) = 0$.

8 Extended Examples

8.1 The Sorites Paradox

Let $H(n)$ be the proposition "the pile of n grains is a heap." Classical logic forces a sharp boundary: there is some k such that $H(k)$ is true and $H(k-1)$ is false, which is intuitively absurd.

In $L_?$ we model the situation as follows:

- $v(H(1)) = 0$ (one grain is not a heap)
- $v(H(10^6)) = 1$ (a million grains is a heap)
- $v(H(k)) = u$ for k in the borderline range

Instead of forcing a decision about borderline cases, we assert $?H(k)$, which has value u regardless of what $v(H(k))$ is. Under $\vdash_q$, $?H(k)$ is acceptable in discourse; under $\vdash_s$, it cannot license a strict inference about the location of a sharp boundary.

The sorites reasoning typically proceeds via Modus Ponens on the premise:

$$H(k) \to H(k-1) \quad \text{("adding one grain cannot make a heap a non-heap")}$$

In $\vdash_s$, if $v(H(k)) = u$, then $v(H(k) \to H(k-1)) = \max(u, v(H(k-1)))$ may also be u, and neither premise reaches value 1. The strict sorites chain breaks down: the inference from the borderline premises to the absurd conclusion requires each link to have value 1, which the u-valued cases prevent.

8.2 Paradoxical Sentences

Consider a Liar-style sentence L such that classical logic forces either $L = 1$ or $L = 0$, both yielding contradiction. Rather than assign L a special status outside the logic, we adopt the policy: place $?L$ in the theory wherever L might appear as a premise or conclusion.

Then $v(?L) = u$ in all valuations. Such sentences do not become theorems of $\vdash_s$ (since they are never valued 1), but they are always designated for $\vdash_q$. Paradoxical sentences are thus given a formal home as permanently open questions—sentences the logic refuses to resolve, without refusing to admit them.

8.3 Scientific Inquiry

Consider a research community investigating the proposition T: "dark energy is a cosmological constant." At the current stage of inquiry:

- The community has not settled T as true or false.
- It would be incorrect to assert T as a strict premise.
- It would be incorrect to dismiss T as meaningless.

The appropriate formal status is $?T$: T is under inquiry, admitted into discourse as a live question, but not licensed to drive strict inferences. Over time, as evidence accumulates, the community might promote $?T$ to T (strict truth), or demote it toward $\neg T$. The ? operator models the epistemic status of an ongoing investigation without prejudging its outcome.

8.4 Pedagogical Context

Consider a student encountering for the first time the statement V: "Euclidean geometry is the one true geometry." A good teacher neither asserts V dogmatically ($V = 1$) nor dismisses it ($V = 0$, meaningless). The teacher holds $?V$ open: Euclidean geometry is a genuine and powerful formal system; whether it is uniquely "true" depends on what "true" means for formal systems. The student is invited to investigate V, not to memorize its truth value. This is the logical form of what elsewhere in the book is called keeping the question open.

9 Comparison with Related Logics

9.1 Strong Kleene Logic K3

K3 is the restriction of $\vdash_s$ to ?-free formulas. It uses truth values $\{1, u, 0\}$ with designated value $\{1\}$ and the same Strong Kleene tables. In K3, the value u represents undefined or indeterminate, but there is no syntactic way to mark which propositions are being held open; the indeterminacy is purely semantic.

$L_?$ adds the ? operator, which makes the act of questioning a first-class syntactic citizen. A sentence can be explicitly placed under inquiry regardless of its underlying semantic value, and the logic tracks this marking through its two consequence relations.

9.2 Priest's Logic of Paradox LP

LP uses the same Kleene truth tables but designates $\{1, u\}$. It was designed to handle paradoxical sentences (particularly the Liar) by allowing them to take value u ("both true and false") without explosion. The consequence relation of LP coincides with our $\vdash_q$ on ?-free formulas.

The difference is conceptual. In LP, the value u represents *gluts* (sentences that are both true and false), and designation of u reflects a paraconsistent stance. In $L_?$, the value u represents *openness* (sentences under inquiry), and designation of u for $\vdash_q$ reflects an inquiry-permissive stance. The same formal structure carries a different philosophical interpretation. The ? operator, absent from LP, allows $L_?$ to distinguish explicitly between formulas that merely happen to have value u and formulas that have been actively placed under inquiry.

Logic	Values	Designated	LEM valid?	? operator
Classical	$\{1,0\}$	$\{1\}$	Yes	No
K3	$\{1,u,0\}$	$\{1\}$	No	No
LP	$\{1,u,0\}$	$\{1,u\}$	Yes	No
$L_?$ ($\vdash_s$)	$\{1,u,0\}$	$\{1\}$	No	Yes
$L_?$ ($\vdash_q$)	$\{1,u,0\}$	$\{1,u\}$	Yes	Yes

9.3 Bochvar's Logic B3

Bochvar (1938) introduced a three-valued logic with an "external" assertion operator that maps u to 0 and 1 to 1. His system was designed to prevent paradoxical sentences from infecting classical reasoning: the external connectives quarantine the indeterminate value.

Our ? operator works in the opposite direction: rather than collapsing u into the safe binary, it deliberately forces the u value, making a sentence into a question. Where Bochvar's assertion operator is a gate that closes off indeterminacy, ? is a gate that opens it.

9.4 Erotetic Logic and Inquisitive Semantics

Wiśniewski's erotetic logic (1995, 2013) develops a formal theory of questions, modeling their evocation and answerhood conditions. Inquisitive semantics (Ciardelli, Groenendijk, Roelofsen 2019) treats questions as sets of possibilities, giving them a compositional semantic treatment alongside statements.

$L_?$ is technically simpler than either of these frameworks: the ? operator is a blunt instrument (constant value u) rather than a structured interrogative meaning. The advantage is tractability and conceptual transparency. The disadvantage is that $L_?$ cannot model the *content* of a question—it cannot distinguish between "Is it raining?" and "Who wrote the Iliad?" Both become $?p$ for some p, differing only in the content of p.

An extension of $L_?$ that interacts with inquisitive semantics would be a natural direction for future work.

10 A Single-Relation Variant: $L_?^*$

For some purposes it is useful to work with a single consequence relation.

Definition 10.1 ($L_?^*$)**.** The language, truth values, and connective semantics are as before. We define a single consequence relation $\vdash$ by:

$$\Gamma \vdash \varphi \iff \text{for every valuation } v, \text{ if } v(\gamma) = 1 \text{ for all } \gamma \in \Gamma, \text{ then } v(\varphi) \in \{1, u\}.$$

Thus $\vdash$ coincides with $\vdash_q$ from $L_?$.

In $L_?^*$, all reasoning is conducted at the inquiry level. The system expresses the fact that, once we adopt the questioning stance, we are no longer in the business of deducing only settled truths. Our consequence relation tracks preservation of non-falsity rather than classical truth.

Remark 10.2. One can recover a strict layer within $L_?^*$ by restricting attention to those formulas that are forced to value 1 in all valuations making Γ true. However, this becomes a metatheoretic notion rather than a separate formal relation.

The single-relation variant is appropriate when one's primary concern is *not generating false conclusions* rather than establishing truths. This is precisely the logical structure of inquiry-as-process: the scientist, teacher, or philosopher who holds questions open is not trying to derive falsehoods; she is trying to preserve the space of possibility until the evidence is in.

11 Conclusion and Further Directions

The logics $L_?$ and $L_?^*$ provide a simple, decidable, well-structured formal apparatus for representing the act of holding a sentence open as a question. The key results are:

- Both consequence relations satisfy Reflexivity, Monotonicity, and Cut (Tarskian structure).
- $\vdash_s \subsetneq \vdash_q$ (strict inclusion, with LEM as witness).
- $\vdash_s$ has no tautologies; $\vdash_q$ recovers LEM and Non-Contradiction.
- Questioned formulas $?\varphi$ are inquiry-valid but not strictly valid; they participate in discourse without functioning as live inferential premises. The proof system enforces this separation via the blocking rule; at the semantic level these formulas yield only vacuous validity.
- $?$ is idempotent, and $\neg ?\varphi$ is also open (questioning cannot be closed by negation).
- The system is sound and decidable.

The key idea underlying all of this is that truth is not the only status worthy of formal attention. The open value u matters in its own right, especially when we see vagueness as the embodiment of inquiry rather than a defect to be eliminated.

Further directions. Many extensions are possible.

1. **Quantifiers.** Adding $\forall$ and $\exists$ would allow us to model inquiry about general claims ("Are all ravens black?") and particular instances.
2. **Refined ? operator.** One could distinguish $?\varphi$ (held open by choice) from $\sim\varphi$ (semantically indeterminate), giving a four-valued system that separates epistemic and ontological sources of openness.
3. **Dynamic inquiry.** A temporal or dynamic logic extension could model the process by which questions are opened and closed, capturing the arc of an inquiry from first question to settled conclusion.
4. **Erotetic connection.** Relating $L_?$ formally to Wiśniewski's erotetic consequence relations would allow the logic to model not just the marking of questions but their evocation and answerhood.
5. **Pedagogical modeling.** The logic was partly motivated by classroom observations (see the companion dissertation *Vagueness as the Embodiment of Inquiry*). A formal model of a student's reasoning under $L_?$—tracking when questions are opened, held, or closed—could give precise meaning to the idea that inquiry, not knowledge, is the proper goal of education.

The present note is intended as a first, workable formalization of the "questioning operator" idea. Its main virtue is simplicity: it gives us a way to treat certain problematic, vague, or paradoxical sentences not as errors to be erased, but as live questions that may accompany our reasoning without destroying it.

Acknowledgment. The ideas in this paper grew out of years of teaching logic to students who were not supposed to question their textbooks, and who were better at it than the textbooks deserved.